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# Bend–stripes in hybrid aligned nematic layers

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The occurrence of bend–stripes also involving twist distortion is investigated in hybrid aligned nematic layers, with stronger tilt-anchoring at the homeotropic substrate than at the planar one. The modulated structure is found to exist above the thickness threshold $d_a$ for the aperiodic bend–splay deformation. The periodicities of the director azimuth and polar angle are incommensurate along the normal to the cell plates. In principle, two independent periodic modes are possible, the first one stable, with threshold $d_p = d_a$, the second one metastable, with threshold $d_p > d_a$, according to the values of the torsional extrapolation lengths. Both transitions are continuous with respect to the in-plane wavenumber. The second mode can appear only if the difference between the torsional and the tilt extrapolation lengths is positive at the planar substrate. Moreover, both thresholds are independent of the saddle-splay elastic constant.



## 1. Introduction

In the past few years a number of authors [1–5] dealt with the occurrence, in hybrid aligned nematics (HAN, see figure 1), of static splay-type periodic patterns (S-PHAN). Such a spatial distortion may appear when the HAN cell is weakly anchored for tilt, the tilt-anchoring being stronger at the unidirectional planar (P-) substrate than at the homeotropic (H-) one.

In fact, the splay-type periodic distortion is energetically favoured, with respect to the aperiodic splay–bend, when the cell thickness, $d$, is such that $d_p < d < d_a$, where $d_p$ is the critical thickness, below which just the P-alignment is allowed, whereas $d_a$ is the critical thickness, above which the aperiodic HAN structure takes place [1, 2]. It is well known [6] that $d_a = \mathscr{L}_{\theta 0} - \mathscr{L}_{\theta 1}$ is dependent only on the tilt extrapolation lengths [7, 8] $\mathscr{L}_{\theta j} \equiv K_{11}/W_j$, where $K_{11}$ is the splay elastic constant and $W_j$ is the tilt anchoring strength at the $j$th wall, according to the Rapini–Papoular model [9, 10]. Instead, $d_p$ was also found to be dependent upon the twist anchoring strengths [2] $L_{\phi j} \equiv K_{22}/W_{\phi j}$, where $K_{22}$ is the twist elastic constant and $W_{\phi j}$ is the twist anchoring strength at the $j$th substrate.

Moreover, $d_p$ exhibits a strong dependence on the saddle-splay elastic constant $K_{24}$ [3], which affects the boundary conditions, due to its surface-like character [11], and on a possible external magnetic field **H** [5].

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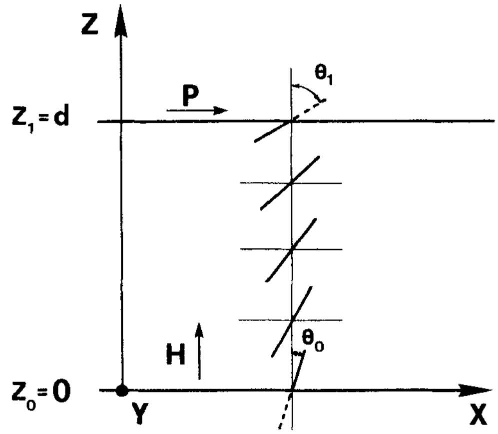


Figure 1. Hybrid aligned nematic (HAN) layer with a tilt anchoring stronger at the substrate $z_0=0$, where the easy direction is homeotropic (H), than at the substrate $z_1=d$, where the easy direction is planar and parallel to the $x$ axis (P). For a layer thickness $d$ greater than a critical one, $d_a$, a planar bend-splay distortion may exist, with $\theta_0<\pi/2-\theta_1$. Such a distortion becomes essentially bend when $d\to d_a$. For $d<d_a$ only an undeformed H-state is allowed.



On the other hand, Allender *et al.* [12] studied the occurrence of bend stripes, due to a magnetic field, in homeotropic nematic films. The field, **H**, was applied in the layer plane, and the anchoring was strong at both walls. In [12], a threshold $\mathbf{H}_p$ was found, slightly greater than the aperiodic Fréedericksz threshold $\mathbf{H}_a$, in contrast with the case of splay-stripes, which may appear with a threshold $\mathbf{H}_p<\mathbf{H}_a$ in P-nematic layers subjected to a magnetic field normal to the cell plates [13, 14]. A result similar to the one in [12] was found by Kini in a bend geometry, obtained in a uniformly tilted nematic cell strongly anchored, by means of a magnetic field perpendicular to the undisturbed director lines [15].

In the present paper we investigate the appearance of a static bend-type periodic distortion in hybrid aligned nematics (B-PHAN, see figure 2). Such a structure may occur in principle, only when the tilt anchoring at the H-wall is stronger than the one at the P-wall. We show that the B-PHAN can appear in two independent modes. The first mode exhibits a threshold $d_d^\theta=d_a$, where $d_a=L_{\theta 1}-L_{\theta 0}$, the tilt extrapolation lengths at the $j$th surface now being $L_{\theta j}\equiv K_{33}/W_{\theta j}$, and $K_{33}$ is the bend elastic constant. The second mode exhibits a threshold $d_p^\phi>d_a$, provided the twist anchoring strength $L_{\phi 1}$ at the P-wall is greater than the tilt anchoring strength $L_{\theta 1}$ at the same wall.

## 2. Theory

In the framework of the first order elastic continuum theory [11], the bulk free energy density is given by

$$f_b=\tfrac{1}{2}\{K_{11}(\operatorname{div}\hat{\mathbf{n}})^2+K_{22}(\hat{\mathbf{n}}\cdot\mathbf{rot}\,\hat{\mathbf{n}})^2+K_{33}(\hat{\mathbf{n}}\wedge\mathbf{rot}\,\hat{\mathbf{n}})^2\}$$
$$-(K_{22}+K_{24})\operatorname{div}[\hat{\mathbf{n}}\cdot\operatorname{div}\hat{\mathbf{n}}+\hat{\mathbf{n}}\wedge\mathbf{rot}\,\hat{\mathbf{n}}] \quad (1)$$

where $\hat{\mathbf{n}}$ is the nematic director.

Let us assume a cartesian frame of reference $[xyz]$, $[xy]$ being the plane coincident with the H-substrate $z_0=0$, whereas $z$ is the coordinate normal to the cell walls. Then

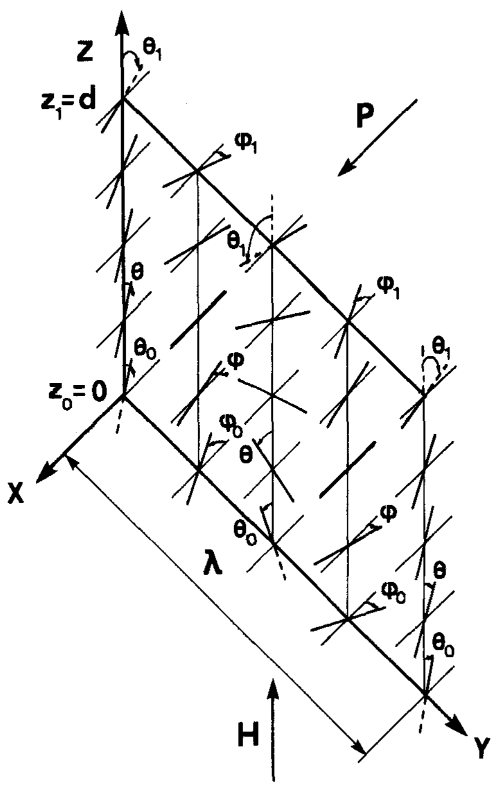


Figure 2. Modulated structure of bend-type in a nematic film with the same boundary conditions as in figure 1 (B-PHan). The parameter $\lambda$ is the distortion wavelength along the $y$ axis, parallel to the cell plates and normal to the aperiodic HAN deformation plane $[zx]$.



the P-substrate is identified by $z_1 = d$. Moreover, at the P-substrate, the easy direction is parallel to the $x$ axis. In order to avoid any degeneration in the boundary conditions when $d \to d_p$ for $d > d_p$, a convenient polar frame of reference has to be chosen, with $[zx]$ as the equatorial plane and $y$ as the polar axis (see figure 3). Hence the azimuth $\theta$ is a tilt angle (which is essentially a bend when $\theta \to 0$), whereas the polar angle $\phi$ is essentially twist only for $\theta \to \pi/2$, and otherwise comprises both twist and tilt, particularly with $\theta \to 0$, $\phi \to 0$, close to the transition between the H-configuration and the B-PHAN. The local director is generally given by

$$\hat{\mathbf{n}} = \hat{\mathbf{i}} \cos\phi \sin\theta + \hat{\mathbf{j}} \sin\phi + \hat{\mathbf{k}} \cos\phi \cos\theta. \tag{2}$$

The surface free energy density due to the anchoring, according to Rapini–Papoular [7, 8] reads, in covariant form

$$f^j_{\mathrm{W}} = [a_j(\hat{\mathbf{n}} \cdot \hat{\mathbf{i}})^2 - b_j(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})^2]\delta \quad (z = jd), \tag{3}$$

with $a_j$, $b_j$ positive at the H-wall ($j = 0$), and negative at the P-wall ($j = 1$), $\delta$ being the Dirac symbol. Thus

$$f^j_{\mathrm{W}} = \tfrac{1}{2}[(-1)^j W_{\theta j} \sin^2\theta_j \cos^2\phi_j + W_{\phi j} \sin^2\phi_j]\delta \quad (z = jd). \tag{4}$$

Note that the tilt anchoring strengths $W_{\theta i}$ are derived by normalizing equation (3) with $\phi_j = 0$, while the torsional anchoring strength $W_{\phi j}$ is normalized by applying to

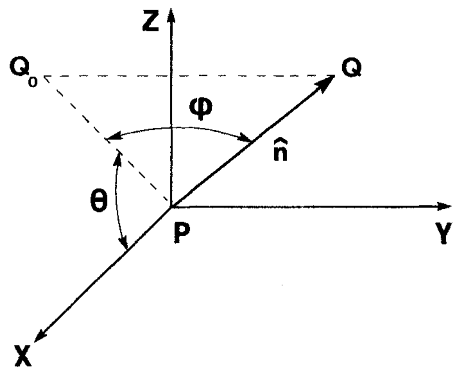


Figure 3. Local polar frame of reference, with equatorial plane [zx] coincident with the HAN distortion plane, and polar axis $y$, parallel to the cell plates. $\theta$ is the azimuth, and $\phi$ is the polar angle related to the director $\hat{\mathbf{n}}$, which is a unit vector defining the average direction of the nematic molecules in a small but macroscopic volume positioned at the arbitrary point $P$.



equation (3) the condition $\theta_j=\pi/2$. In fact we have

$$\left.\begin{aligned} a_j&=\tfrac{1}{2}W_{\phi j},\\ b_j&=\tfrac{1}{2}[W_{\theta j}+(-1)^i W_{\phi j}]\end{aligned}\right\} \tag{5}$$

and in the limit $\phi\to 0$, $\theta\to 0$ equation (4) reduces to

$$f^j_{\mathrm{W}}=\tfrac{1}{2}[(-1)^j W_{\theta j}(\theta_j^2+\phi_j^2)+W_{\phi j}\phi_j^2]\delta \quad (z=jd). \tag{6}$$

In equation (6) the ambiguous character, tilt and twist, of the polar angle $\phi_j$ is pointed out. Note that the energetic contribution of the surface tilt deformation is positive at the H-wall, and negative at the other one.

Let us consider the free energy $F$ of a reference volume of nematic layer, having length $d_x$ along the $x$ axis, width $\lambda$ equal to the stripes wavelength along the $y$ axis, and height along the $z$ axis equal to the film thickness $d$. By looking for the transition between the undeformed H-state and the B-PHAN structure, where $d$ plays the role of an external field, we can linearize the reduced free energy $G=2Fd_x/K_{22}$. Thus we have up to the second order in $\theta$, $\phi$ and in their derivatives

$$\left.\begin{aligned} G&=G_{\mathrm{b}}+G_{\mathrm{s}},\\ G_{\mathrm{b}}&=\int_0^\lambda dy\int_0^d \{\kappa_1\phi_y^2+\theta_y^2+\kappa_3(\phi_z^2+\theta_z^2)\}\,dz,\\ G_{\mathrm{s}}&=\int_0^\lambda dy\{-\kappa_3L_{\theta 1}^{-1}(\theta_1^2+\phi_1^2)+L_{\phi 1}^{-1}\phi_1^2+\kappa_3L_{\theta 0}^{-1}(\theta_0^2+\phi_0^2)\\ &\quad+L_{\phi 0}^{-1}\phi_0^2-2(1+\kappa_4)[\phi_{y1}-\phi_{y0}]\},\end{aligned}\right\} \tag{7}$$

$G_{\mathrm{b}}$, $G_{\mathrm{s}}$ being bulk and surface contribution, respectively.

In system (7), $\kappa_1\equiv K_{11}/K_{22}$, $\kappa_3\equiv K_{33}/K_{22}$ are the bulk elastic ratios, $\kappa_4\equiv K_{24}/K_{22}$ is the surface-like elastic ratio, and the subscripts $y$ and $z$ mean the derivatives with respect to $y$ and $z$ themselves. Hence, the Euler–Lagrange equations are uncoupled homogeneous second order equations

$$\left.\begin{aligned}\theta_{yy}+\kappa_3\theta_{zz}&=0,\\ \phi_{yy}+\kappa_{\rm BS}\phi_{zz}&=0,\end{aligned}\right\}\tag{8}$$

where $\kappa_{\rm BS}\equiv K_{33}/K_{11}$ is the bulk bend–splay elastic ratio. The boundary conditions are to be derived from

$$\left.\begin{aligned}-\left(\frac{\partial g_{\rm b}}{\partial \chi_z}\right)_0+\frac{\partial g_{\rm s}}{\partial \chi_0}&=0,\\ \left(\frac{\partial g_{\rm b}}{\partial \chi_z}\right)_1+\frac{\partial g_{\rm s}}{\partial \chi_1}&=0,\end{aligned}\right\}\tag{9}$$

where $g_{\rm b}$, $g_{\rm s}$ are the kernels of $G_{\rm b}$, $G_{\rm s}$ and $\chi$ assumes the values of either the azimuth $\theta$ or the polar angle $\phi$. Thus the linearized boundary conditions can be written as

$$\left.\begin{aligned}\theta_{z0}-L_{\theta 0}^{-1}\theta_0&=0,\\ \theta_{z1}-L_{\theta 1}^{-1}\theta_1&=0,\\ \phi_{z0}-(L_{\theta 0}^{-1}+L_{\phi 0}^{-1})\phi_0&=0,\\ \phi_{z1}-(L_{\theta 1}^{-1}-L_{\phi 1}^{-1})\phi_1&=0.\end{aligned}\right\}\tag{10}$$

We stress the fact that the boundary conditions are also uncoupled. Hence, two periodic modes having, in principle, different thresholds are expected, the first depending only on the tilt extrapolation lengths, the second one depending also on the twist extrapolation lengths. In addition, note that system (10) is independent of $K_{24}$: this means that the threshold of B-PHAN is not affected by the surface-like elasticity, in contrast with the behaviour of the S-PHAN configuration [3]. In fact, looking for periodic solutions of the type

$$\left.\begin{aligned}\theta&=\Theta\exp[{\rm i}(\alpha z+qy)],\\ \phi&=\Phi\exp[{\rm i}(\beta z+qy)],\end{aligned}\right\}\tag{11}$$

when the $y$ wavenumber $q=2\pi/\lambda$ is given, we derive

$$\left.\begin{aligned}\alpha&=\pm\frac{iq}{\sqrt{\kappa_3}},\\ \beta&=\pm\frac{iq}{\sqrt{\kappa_{\rm BS}}}.\end{aligned}\right\}\tag{12}$$

Thus the solutions read

$$\left.\begin{aligned}\theta&=\left[A\cosh\left(\frac{qz}{\sqrt{\kappa_3}}\right)+B\sinh\left(\frac{qz}{\sqrt{\kappa_3}}\right)\right]\cos qy,\\ \phi&=\left[C\cosh\left(\frac{qz}{\sqrt{\kappa_{\rm BS}}}\right)+D\sinh\left(\frac{qz}{\sqrt{\kappa_{\rm BS}}}\right)\right]\sin qy\end{aligned}\right\}\tag{13}$$

and the periodicities of $\theta$ and $\phi$ are found to be incommensurate, as expected. By substituting system (13) into the uncoupled boundary conditions (10), two independent determinants must vanish, in order to avoid trivial solutions, thus giving two dispersion relations of the type

$$d=\frac{\sqrt{\kappa}}{q}\operatorname{arc\,tanh}\left\{\frac{q}{\sqrt{\kappa}}\frac{m_0-m_1}{m_0m_1}-q^2/\kappa\right\}, \tag{14}$$

where $m\equiv L_{\theta j}^{-1}$ and $\kappa\equiv\kappa_3$ for the dispersion relation concerned with the behaviour of the azimuth $\theta$, whereas $m_0\equiv L_{\theta 0}^{-1}+L_{\phi 0}^{-1}$, $m_1\equiv L_{\theta 1}^{-1}-L_{\phi 1}^{-1}$ and $\kappa\equiv\kappa_{\rm BS}$ for the other dispersion relation, coming from the polar angle $\phi$. The tilt anchoring stronger at the H-side ensures $m_0>m_1$.

Equation (14) defines the layer thickness $d$ as a function of $q$ in the range $[0,m_1]$. Note that $m_1$ must be positive, since both $d$ and $q$ are positive, as defined. Thus, the mode relevant to the behaviour of the polar angle $\phi$ may exist only if $L_{\phi 1}>L_{\theta 1}$. The minima of the two functions $d(q)$ are the thresholds $d_{\rm p}^{\theta}$, $d_{\rm p}^{\phi}$ for the B-PHAN modes, which exist for $d>d_{\rm p}$, whereas for $d<\min\{d_{\rm p}^{\theta},d_{\rm p}^{\phi}\}$ only the undeformed H-state is allowed. Here $q$ plays the role of an order parameter.

Since both functions $d(q)$ are monotonically increasing with $q$ (see figure 4), the two minima are coincident with $q=0$, giving

$$\left.\begin{aligned} d_{\rm p}^{\theta}&=L_{\theta 1}-L_{\theta 0},\\ d_{\rm p}^{\phi}&=\frac{L_{\theta 1}L_{\phi 1}}{L_{\phi 1}-L_{\theta 1}}-\frac{L_{\theta 0}L_{\phi 0}}{L_{\phi 0}+L_{\theta 0}}\geqslant d_{\rm p}^{\theta},\end{aligned}\right\} \tag{15}$$

and at the thresholds the value of the B-stripes wavelength is $\lambda_{\rm p}=+\infty$. Hence, the thresholds (15) are characterized by continuous transitions between the H-state and the B-PHAN states.

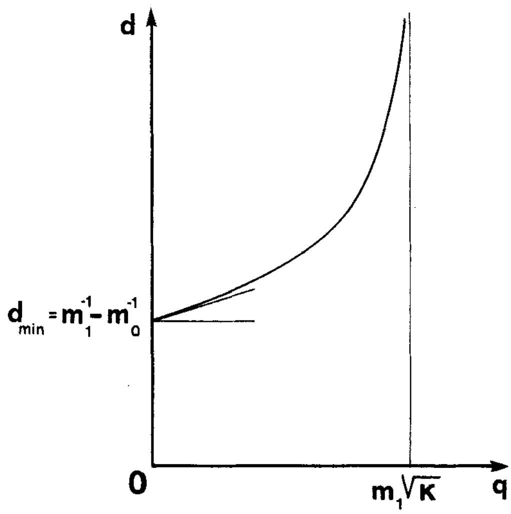


Figure 4. Behaviour of the function $d$ versus $q$, where $q=2\pi/\lambda$ is the wavenumber of the B-PHAN stripes long the $y$ axis. The considered function is monotonically increasing in all the range $(0,m_1)$, with the first derivative positive everywhere. Thus the absolute minimum is reached for $q=0$, and its value is $d_{\rm min}=m_1^{-1}-m_0^{-1}$. Such a condition provides the existence of two B-PHAN modes, with thresholds $d_{\rm p}^{\theta}=d_{\rm a}$, $d_{\rm p}^{\phi}\geqslant d_{\rm a}$. The first mode is characterized by $m_j=L_{\theta j}^{-1}$, the second one by $m_0=L_{\theta 0}^{-1}+L_{\phi 0}^{-1}$, $m_1=L_{\theta 1}^{-1}-L_{\phi 1}^{-1}$. Both modes exhibit continuous transition coming from the undeformed H-state.

It is interesting to perform a linear analysis of the stability of the two B-PHAN configurations close to the corresponding thresholds $d_p^\theta$, $d_p^\phi$. The procedure requires the expansion of the reduced free energy $G$ in (7) up to the second order in $q \to q_p = 0$, by assuming in system (13)

$$\left.\begin{aligned} \theta &= (A + B'z)(1 - q^2y^2/2), \\ \phi &= (C + D'z)qy. \end{aligned}\right\}$$

The minima of the quadratic form $G(A, B', C, D')$ for $q = q_p$ are to be investigated. The vanishing of the first derivatives provides the threshold values (15), whereas the $(4 \times 4)$ matrix of the second derivatives gives as principal determinants

$$\left.\begin{aligned} D_{11} &\equiv 2\kappa_3 \lambda d(1 - L_{\theta 1}^{-1} d), \\ D_{22} &\equiv 4\kappa_3^2 \lambda^2 \begin{vmatrix} d(1 - L_{\theta 1}^{-1} d) & L_{\theta 1}^{-1} d \\ L_{\theta 1}^{-1} d & L_{\theta 0}^{-1} - L_{\theta 1}^{-1} \end{vmatrix}, \\ D_{33} &= 0, \\ D_{44} &= 0. \end{aligned}\right\} \quad (16)$$

By assuming $d = d_p^\theta$, system (16) provides $D_{11} > 0$, $D_{22} = 0$: thus the first mode of the B-PHAN structure is stable for $d > d_p^\theta$ close to $d_p^\theta$. Instead, by imposing $d = d_p^\phi$, system (16) still gives $D_{11} > 0$, but $D_{22} < 0$. Hence the second mode of the B-PHAN state is metastable.

Furthermore, we note that the value of $d_p^\phi$ obtained in (15) is just an estimate, if $d_p^\phi > d_p^\theta$. A non-linear analysis should be performed, since $d_p^\phi$ describes a transition between two distorted states: either HAN⇔B-PHAN II, or B-PHAN I⇔B-PHAN II.



## 3. Conclusion

The appearance and the stability of static, bend-type periodic deformations in hybrid aligned nematic films (B-PHAN) have been investigated. Remarkably, two independent B-PHAN modes were found, with thresholds $d_p^\theta = d_a$, where $d_a$ is the threshold for the aperiodic HAN state (I mode), and $d_p^\phi \geqslant d_a$ (II mode). The transitions between the H-state and the B-PHAN structure are continuous. No B-PHAN can exist when $d < d_a$, where the homeotropic undeformed configuration takes place. This result is similar to the one found by Allender *et al.* [12], in a nematic H-layer subjected to a transverse in-plane magnetic field, providing a periodic Fréedericksz transition.

## References


[1] LAVRENTOVICH, O. D., and PERGAMENSHCHIK, V. M., 1990, *Molec. Crystals liq. Crystals*, **179,** 125.

[2] SPARAVIGNA, A., KOMITOV, L., STEBLER, B., and STRIGAZZI, A., 1991, *Molec. Crystals liq. Crystals*, **207,** 265.

[3] SPARAVIGNA, A., and STRIGAZZI, A., 1992, *Molec. Crystals liq. Crystals*, **221,** 109.

[4] SPARAVIGNA, A., KOMITOV, L., and STRIGAZZI, A., 1992, *Molec. Crystals liq. Crystals*, **212,** 289.

[5] SPARAVIGNA, A., KOMITOV, L., LAVRENTOVICH, O. D., and STRIGAZZI, A., 1992, *J. Phys., France*, **2,** 1881.

[6] BARBERO, G., and BARBERI, R., 1983, *J. Phys., France*, **44,** 609.

[7] DE GENNES, P. G., 1974, *The Physics of Liquid Crystals* (Clarendon Press),

[8] KLEMAN, M., 1977, *Point, Lignes, Parois* (Ed. Physique).
[9] RAPINI, A., and PAPOULAR, M., 1969, *J. Phys., France*, **30-CA,** 54.
[10] BARBERO, G., and DURAND, G., 1986, *J. Phys., France*, **47,** 2129.
[11] BARBERO, G., SPARAVIGNA, A., and STRIGAZZI, A., 1990, *Nuovo Cim.* D, **12,** 1259.
[12] ALLENDER, D. W., HORNREICH, R. M., and JOHNSON, D. L., 1987, *Phys. Rev. Lett.*, **59,** 2654.
[13] LONGBERG, F., and MEYER, R. B., 1985, *Phys. Rev. Lett.*, **55,** 718.
[14] MIRALDI, E., OLDANO, C., and STRIGAZZI, A., 1986, *Phys. Rev.* A, **34,** 4348.
[15] KINI, U. D., 1990, *Liq. Crystals*, **2,** 185.